Lecture 16

June 4, 2016

1 The power-law lens

The power-law lens has a convergence profile of the kind

$$\kappa(x) = \frac{3-n}{2} x^{1-n}$$

The corresponding mass profile is

$$m(x) = x^{3-n}$$

which implies that the deflection angle is

$$\alpha(x) = x^{2-n}$$

and that the shear profile is

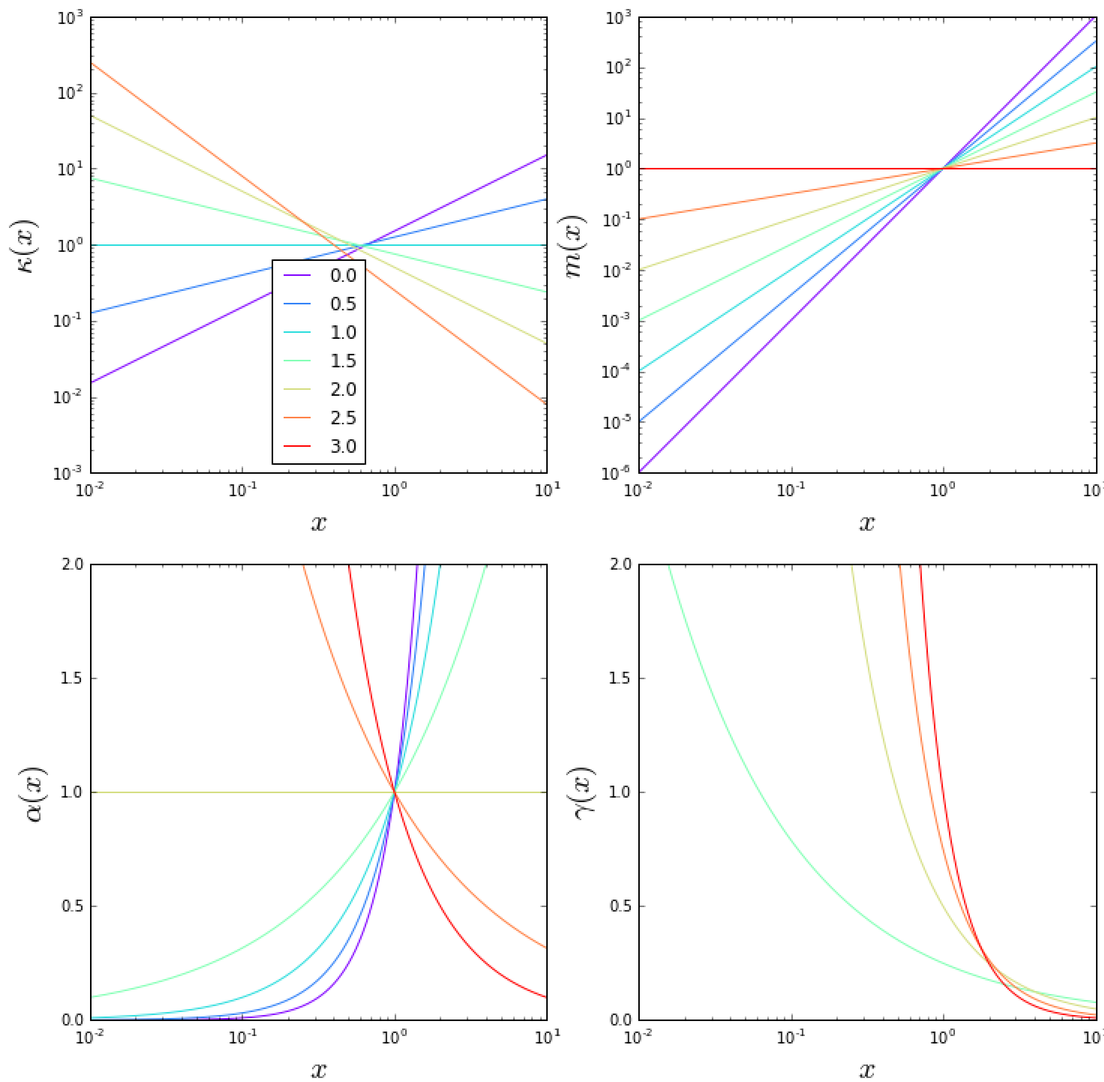
$$\gamma(x) = \frac{m(x)}{x^2} - \kappa(x) = \frac{n-1}{2} x^{1-n} \ .$$

Here are some plots of some of the relevant quantities.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
from matplotlib.pyplot import cm
fig,ax=plt.subplots(2,2,figsize=(12,12))

x=np.logspace(-2,1.0,1000)
n=np.linspace(0,3,7)

def kappa(x,n):
    return((3-n)/2.0*np.abs(x)**(1-n))

def gamma(x,n):
    return((n-1)/2.0*np.abs(x)**(1-n))

def mass(x,n):
    return(np.abs(x)**(3-n))

def alpha(x,n):
    return(np.sign(x)*np.abs(x)**(2-n))

color=iter(cm.rainbow(np.linspace(0,1,n.size)))
ax[0,0].set_xscale('log')
ax[0,0].set_yscale('log')
ax[0,1].set_xscale('log')
ax[0,1].set_yscale('log')
ax[1,0].set_xscale('log')
```

```
    #ax[1,0].set_yscale('log')
    ax[1,1].set_xscale('log')
    #ax[1,1].set_yscale('log')
    ax[1,0].set_ylim([0,2])
    ax[1,1].set_ylim([0,2])
    for i in range(n.size):
        c=next(color)
        ka=kappa(x,n[i])
        m=mass(x,n[i])
        a=alpha(x,n[i])
        g=gamma(x,n[i])
        ax[0,0].plot(x,ka,color=c,label=str(n[i]))
        ax[0,1].plot(x,m,color=c,label=str(n[i]))
        ax[1,0].plot(x,a,color=c,label=str(n[i]))
        ax[1,1].plot(x,g,color=c,label=str(n[i]))

    ax[0,0].legend(loc='best')
    ax[0,0].set_xlabel('$x$',fontsize=20)
    ax[0,0].set_ylabel('$\kappa(x)$',fontsize=20)
    ax[0,1].set_xlabel('$x$',fontsize=20)
    ax[0,1].set_ylabel('$m(x)$',fontsize=20)
    ax[1,0].set_xlabel('$x$',fontsize=20)
    ax[1,0].set_ylabel(r'$\alpha(x)$',fontsize=20)
    ax[1,1].set_xlabel('$x$',fontsize=20)
    ax[1,1].set_ylabel(r'$\gamma(x)$',fontsize=20)
```

```
Out[1]: <matplotlib.text.Text at 0x10fa3f910>
```



1.1 Magnification and critical lines

The magnification is

$$\det A = (1 - \kappa - \gamma)(1 - \kappa + \gamma)$$

In particular, the inverse tangential and the radial magnifications are

$$\lambda_t = 1 - x^{1-n}$$

and

$$\lambda_r = 1 - (2 - n)x^{1-n}$$

The critical lines are given by

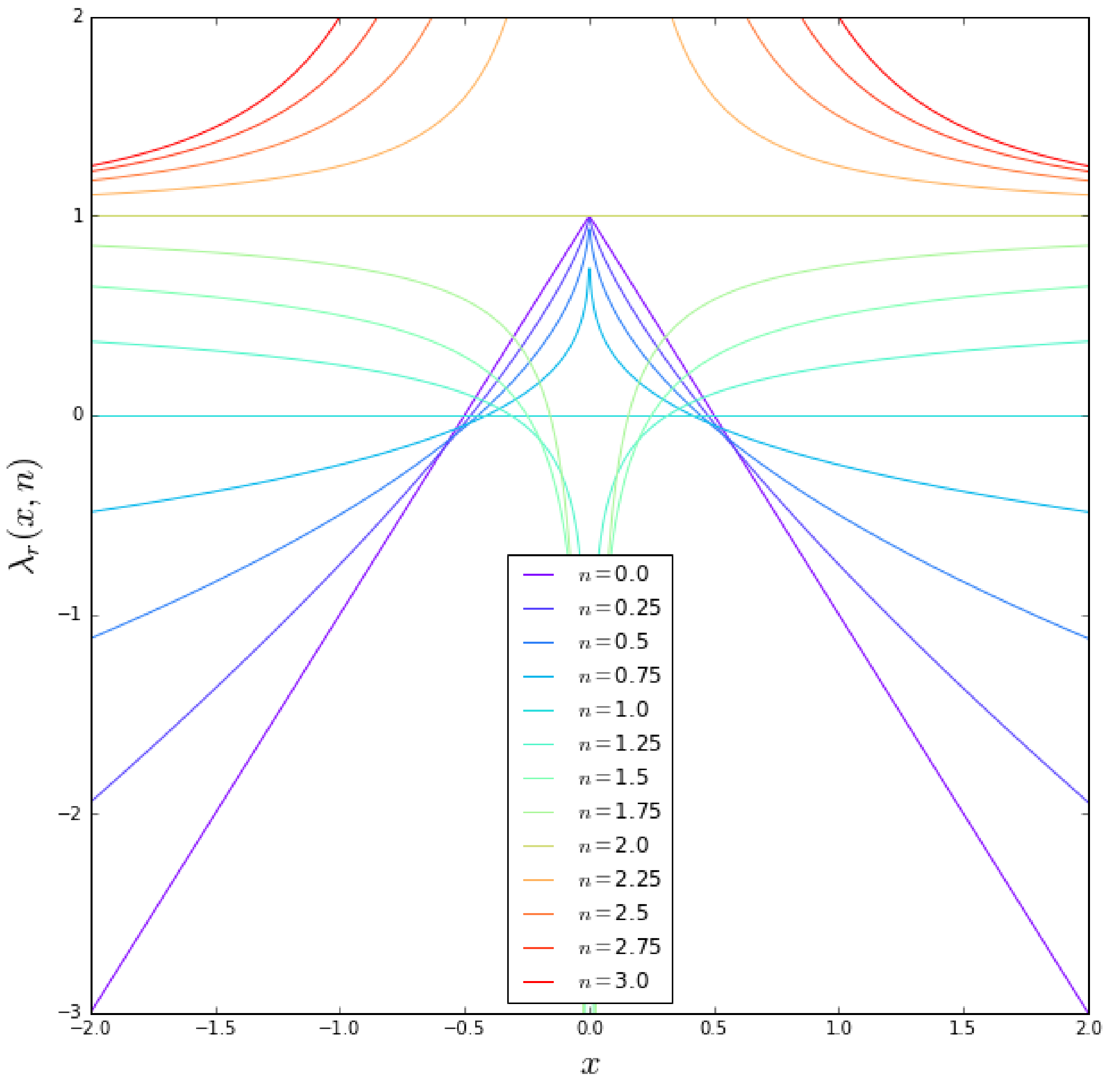
$$x_t = 1$$

and

$$x_r = \left(\frac{1}{2-n}\right)^{\frac{1}{1-n}} = (2-n)^{\frac{1}{n-1}}$$

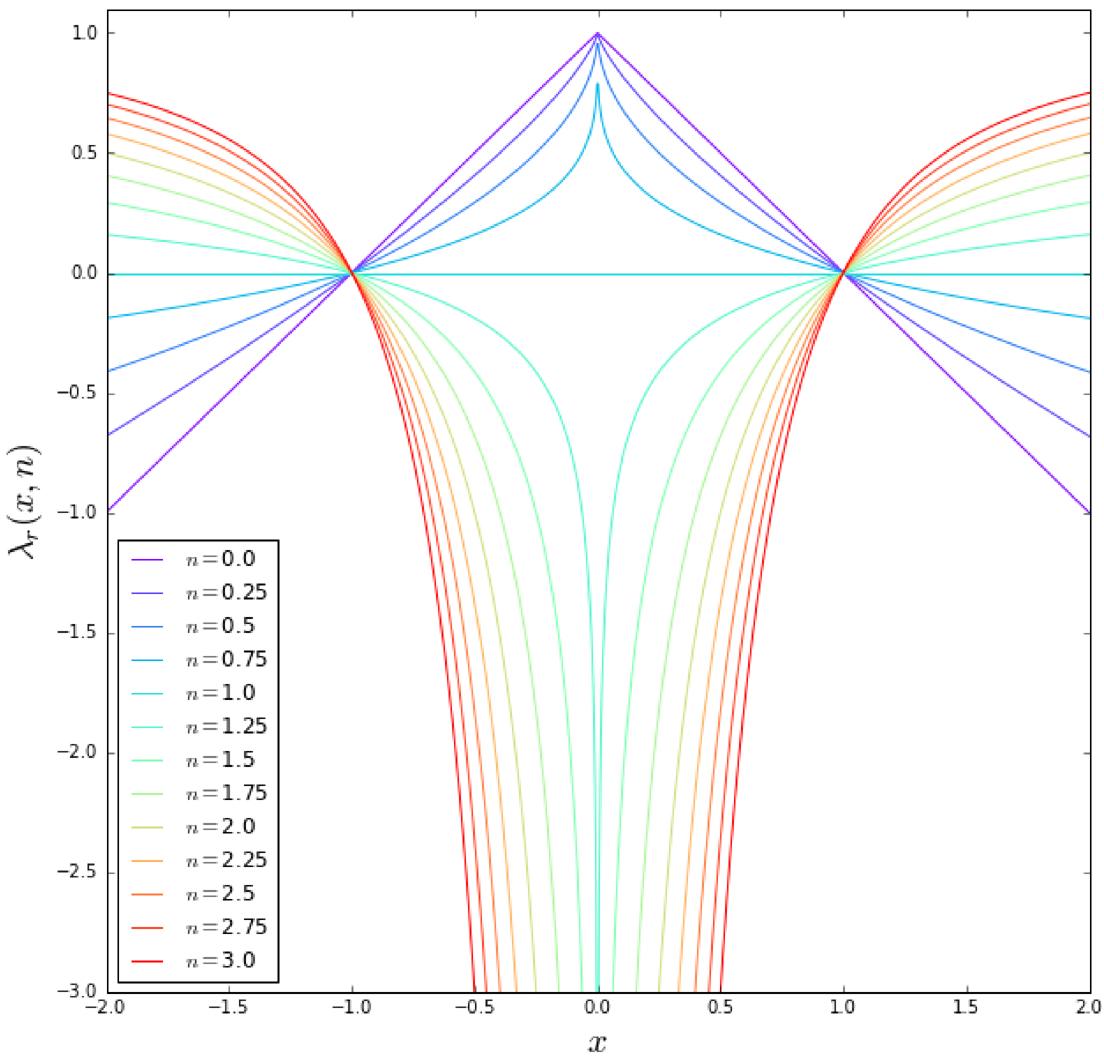
In [2]:
```
n=np.linspace(0.0,3.0,13)
x=np.linspace(-2.0,2.0,1000)


def lambdar_pow(x,n):
    return 1-(2.0-n)*np.abs(x)**(1.0-n)

fig,ax=plt.subplots(1,1,figsize=(10,10))
color=iter(cm.rainbow(np.linspace(0,1,n.size)))

for i in range(n.size):
    c=next(color)
    ax.plot(x,lambdar_pow(x,n[i]),color=c,label=r'$n=$'+str(n[i]))

ax.set_xlabel(r'$x$',fontsize=20)
ax.set_ylabel(r'$\lambda_r(x,n)$',fontsize=20)
ax.legend(loc='best')
ax.set_ylim([-3,2])
```

Out[2]:
```
(-3, 2)
```




```
In [3]: n=np.linspace(0.0,3.0,13)

        def lambdat_pow(x,n):
            return 1-np.abs(x)**(1.0-n)

        fig,ax=plt.subplots(1,1,figsize=(10,10))
        color=iter(cm.rainbow(np.linspace(0,1,n.size)))

        for i in range(n.size):
            c=next(color)
            ax.plot(x,lambdat_pow(x,n[i]),color=c,label=r'$n=$'+str(n[i]))

        ax.set_xlabel(r'$x$',fontsize=20)
        ax.set_ylabel(r'$\lambda_r(x,n)$',fontsize=20)
        ax.legend(loc='best')
        ax.set_ylim([-3,1.1])
```

Out[3]: (-3, 1.1)

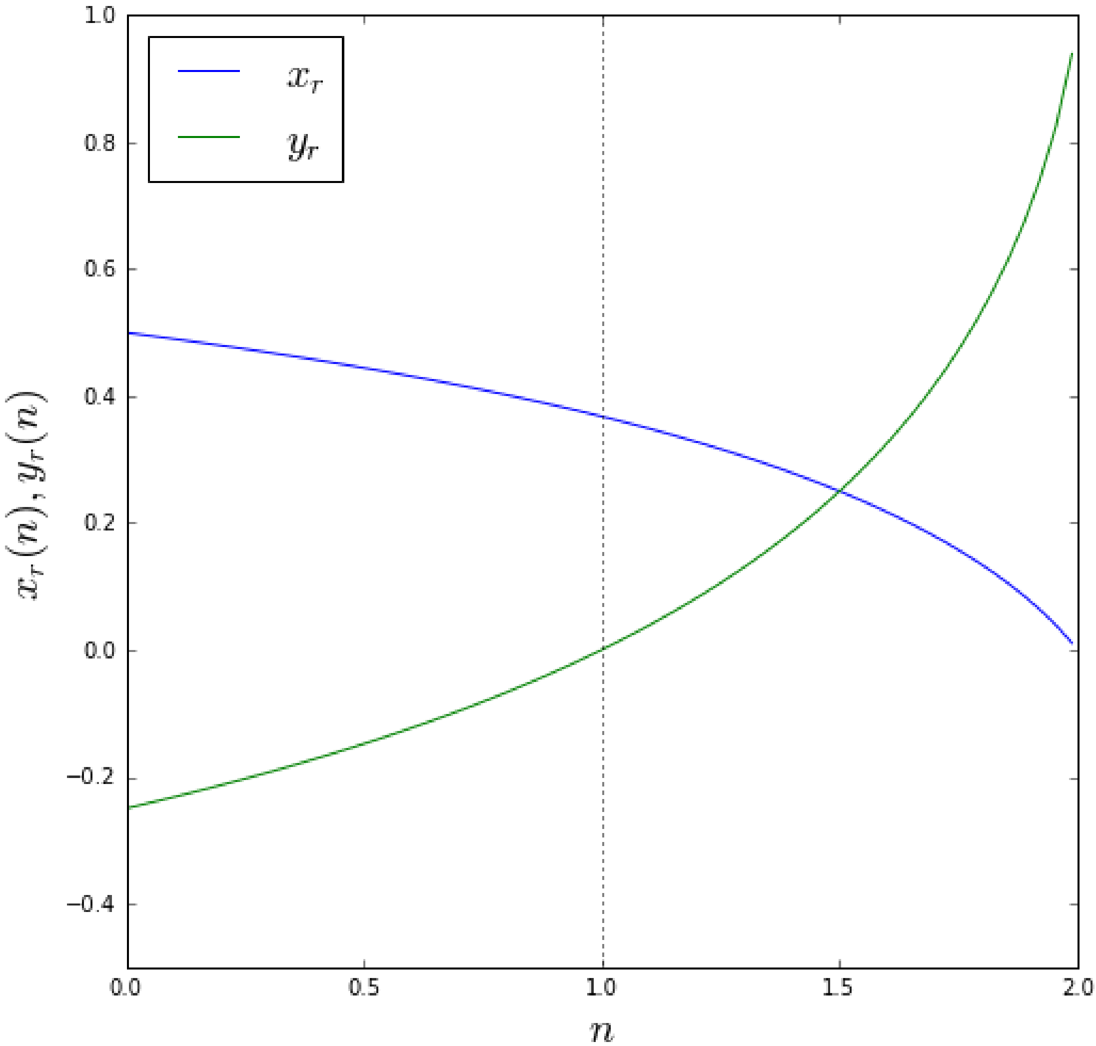


In [21]:
```
n=np.linspace(0.0000001,3,90)

def radcl(n):
    return((2.0-n)**(1./(n-1)))

def radcau(n):
    return -(radcl(n)-alpha(radcl(n),n))



rl=radcl(n)
rc=radcau(n)
fig,ax=plt.subplots(1,1,figsize=(8,8))
ax.plot(n,rl,label=r'$x_r$')
ax.plot(n,rc,label=r'$y_r$')
```

```
ax.set_xlabel('$n$',fontsize=20)
ax.set_ylabel('$x_r(n),y_r(n)$',fontsize=20)
ax.plot([1.0,1.0],[-1,1],':',color='black')
ax.set_ylim([-0.5,1.0])
ax.legend(loc='best',fontsize=20)
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:4: RuntimeWarning
```

```
Out[21]: <matplotlib.legend.Legend at 0x1161d0e10>
```



The radial critical line only exists if $n < 2$. For $n > 2$ there is no radial critical line.

2 Multiple images

2.1 Lenses with $n < 2$

Let's discuss the occurrence of multiple images. If $n < 2$, the deflection angle is zero at the origin. In this case, as it emerges from the image diagram, multiple images can form only if $dy/dx < 0$ somewhere in the lens. This translates into a condition on the derivative of the deflection angle:

$$\frac{d\alpha}{dx} > 1$$

Being $\alpha(x) = x^{2-n}$, we have that

$$\frac{d\alpha}{dx} = (2-n)x^{1-n} \ ,$$

which means that $d\alpha/dx > 1$ for

$$x < (2-n)^{\frac{1}{n-1}} = x_r \ ,$$

in case $n > 1$, and

$$x > x_r$$

otherwise.

Therefore, for $n < 2$, there are multiple images. How many? For $y = y_r$, $d\alpha/dx = 1$ meaning that that the line $x - y_r$ is tangent to $\alpha(x)$ in x_r. For $n > 1$, $y < y_r$, there are three images of the source. Otherwise, only one. In $y = y_r$, two images merge.

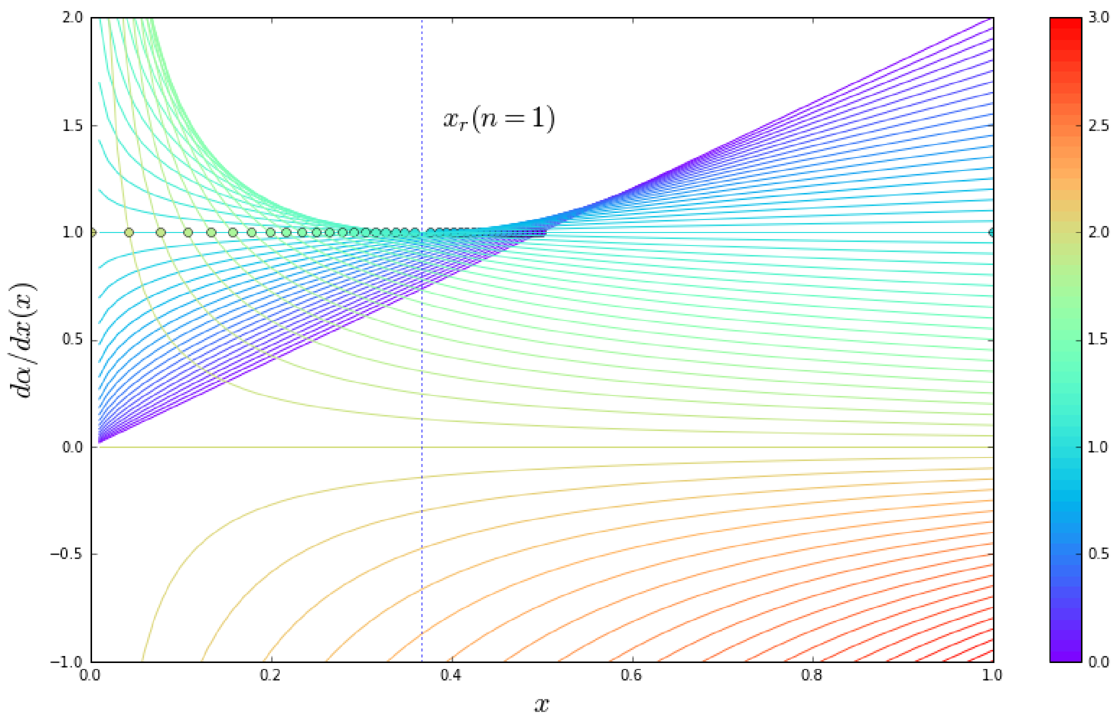
```
In [22]: def dadx(x,n):
             return((2.0-n)*x**(1.0-n))

         x=np.linspace(0.01,1,100)
         n=np.linspace(0,3.0,61)

         #xm=xmax(n)

         color=iter(cm.rainbow(np.linspace(0,1,n.size)))

         Z = [[0,0],[0,0]]
         levels = n#range(min,max+step,step)
         CS3 = plt.contourf(Z, levels, cmap=cm.rainbow)
         plt.clf()

         fig,ax=plt.subplots(1,1,figsize=(14,8))
         ax.set_ylim([-1,2])



         #ax.plot(n,xm,'-')
         xr=radcl(n)
         yr=np.zeros(n.size)+1.0
         for i in range(n.size):
             c=next(color)
             dd=dadx(x,n[i])
             ax.plot(x,dd,'-',color=c)
             ax.plot(xr[i],yr[i],'o',markeredgecolor='black',markersize=6,c=c)

         xr=radcl(1.001)
         xx=[xr,xr]
         yy=[-1.0,2.0]
         #ax.plot(x,dd,'--',color='black')
         ax.plot(xx,yy,':')

         cb=plt.colorbar(CS3)
         cb.set_ticks([0.0,0.5,1.0,1.5,2.0,2.5,3.0])
```

```
ax.set_xlabel('$x$',fontsize=20)
ax.set_ylabel(r'$d\alpha/dx(x)$',fontsize=20)

ax.text(0.39,1.5,'$x_r(n=1)$',fontsize=20)
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:4: RuntimeWarning
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:4: RuntimeWarning
```

Out[22]: <matplotlib.text.Text at 0x118b134d0>

<matplotlib.figure.Figure at 0x114dc85d0>

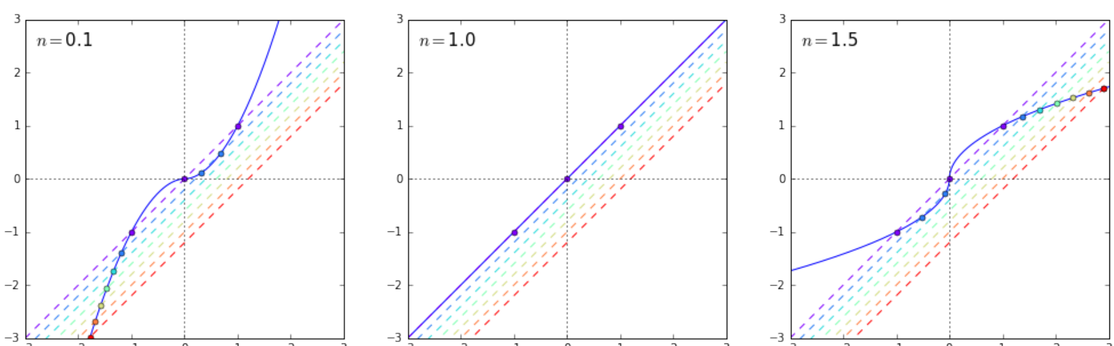


The following are some image diagrams for lenses with $n = 0.1, 1.0, 1.5$

```
In [6]: import math
        from scipy.optimize import *

        n=[0.1,1.0,1.5]

        fig,ax=plt.subplots(1,3,figsize=(17,5))

        x=np.linspace(-3,3,1000)

        def f(x,y):
            return(x-y)

        ys=np.linspace(0,1.2,7)
```

```
    for i in range(len(n)):
        a=alpha(x,n[i])
        ax[i].plot(x,a,'-')
        color=iter(cm.rainbow(np.linspace(0,1,len(ys))))
        for j in range(len(ys)):
            c=next(color)
            fu=f(x,ys[j])
            ax[i].plot(x,fu,'--',c=c)
            def func(xx):
                return f(xx,ys[j])-alpha(xx,n[i])
            x0 = fsolve(func,1.0)
            x1 = fsolve(func,-1.0)
            x3 = fsolve(func,0.0)
            if (np.abs(func(x0))<1e-8):
                ax[i].plot(x0,alpha(x0,n[i]),'o',markersize=5,c=c)
            if (np.abs(func(x1))<1e-8):
                ax[i].plot(x1,alpha(x1,n[i]),'o',markersize=5,c=c)
            if (np.abs(func(x3))<1e-8):
                ax[i].plot(x3,alpha(x3,n[i]),'o',markersize=5,c=c)


        ax[i].set_xlim([-3,3])
        ax[i].set_ylim([-3,3])
        ax[i].text(-2.8,2.5,'$n=$'+str(n[i]),fontsize=15)


        xa=[0.0,0.0]
        ya=[-10,10]
        ax[i].plot(xa,ya,':',color='black')

        xa=[-10.0,10.0]
        ya=[0,0]
        ax[i].plot(xa,ya,':',color='black')
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/scipy/optimize/minpack.py:236: RuntimeW
  improvement from the last ten iterations.
  warnings.warn(msg, RuntimeWarning)
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/scipy/optimize/minpack.py:236: RuntimeW
  improvement from the last five Jacobian evaluations.
  warnings.warn(msg, RuntimeWarning)
```



Some noticeable properties: * these lenses produce three or one image * three images are formed if the source lays inside the radial critical line, i.e. for $y < y_r$, where y_r corresponds to $x_r : dy/dx = 0$

3 Lenses with $n > 2$

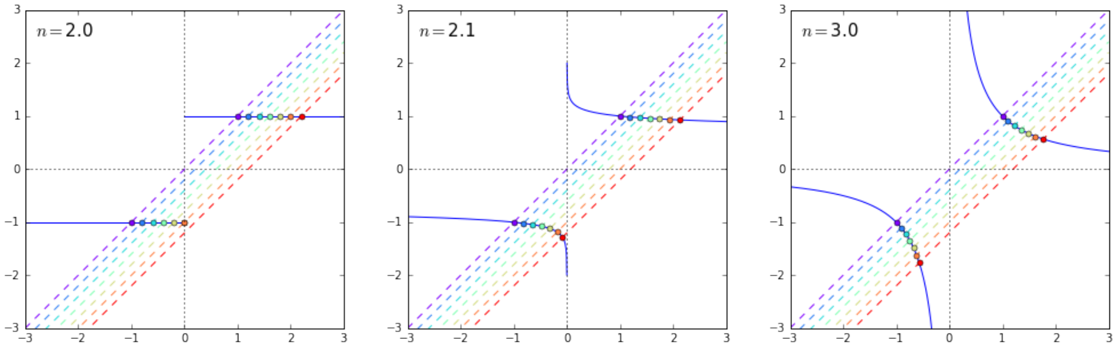
These lenses always produce multiple images because of the singuarity of their lensing potential.

The following are some examples of image diagrams for $n = 2.0, 2.1, 3.0$.

In [7]:
```
n=[2.0,2.1,3.0]

fig,ax=plt.subplots(1,3,figsize=(17,5))

x0=np.linspace(-3,-1.e-3,500)
x1=np.linspace(1.e-3,3,500)

ys=np.linspace(0,1.2,7)

for i in range(len(n)):
    a0=alpha(x0,n[i])
    a1=alpha(x1,n[i])
    ax[i].plot(x0,a0,'-',color='blue')
    ax[i].plot(x1,a1,'-',color='blue')
    color=iter(cm.rainbow(np.linspace(0,1,len(ys))))
    for j in range(len(ys)):
        c=next(color)
        fu=f(x,ys[j])
        ax[i].plot(x,fu,'--',c=c)
        def func(xx):
            return f(xx,ys[j])-alpha(xx,n[i])
        x0_ = fsolve(func,1.0)
        x1_ = fsolve(func,-1.0)
        x3_ = fsolve(func,0.0)
        if (np.abs(func(x0_))<1e-8):
            ax[i].plot(x0_,alpha(x0_,n[i]),'o',markersize=5,c=c)
        if (np.abs(func(x1_))<1e-8):
            ax[i].plot(x1_,alpha(x1_,n[i]),'o',markersize=5,c=c)
        if (np.abs(func(x3_))<1e-8 and j>0):
            ax[i].plot(x3_,alpha(x3_,n[i]),'o',markersize=5,c=c)

    ax[i].set_xlim([-3,3])
    ax[i].set_ylim([-3,3])
    ax[i].text(-2.8,2.5,'$n=$'+str(n[i]),fontsize=15)

    xa=[0.0,0.0]
    ya=[-10,10]
    ax[i].plot(xa,ya,':',color='black')

    xa=[-10.0,10.0]
    ya=[0,0]
    ax[i].plot(xa,ya,':',color='black')
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:20: RuntimeWarning
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:20: RuntimeWarning
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:20: RuntimeWarning
```



Notice that * none of these lenses ever has a central image * while for a lens with $n = 2$ produces multiple images only for $|y| < y_{cut} = 1$, all lenses with $n > 2$ always have 2 images * the x_- image always moves towards the center of the lens as $y \to \infty$

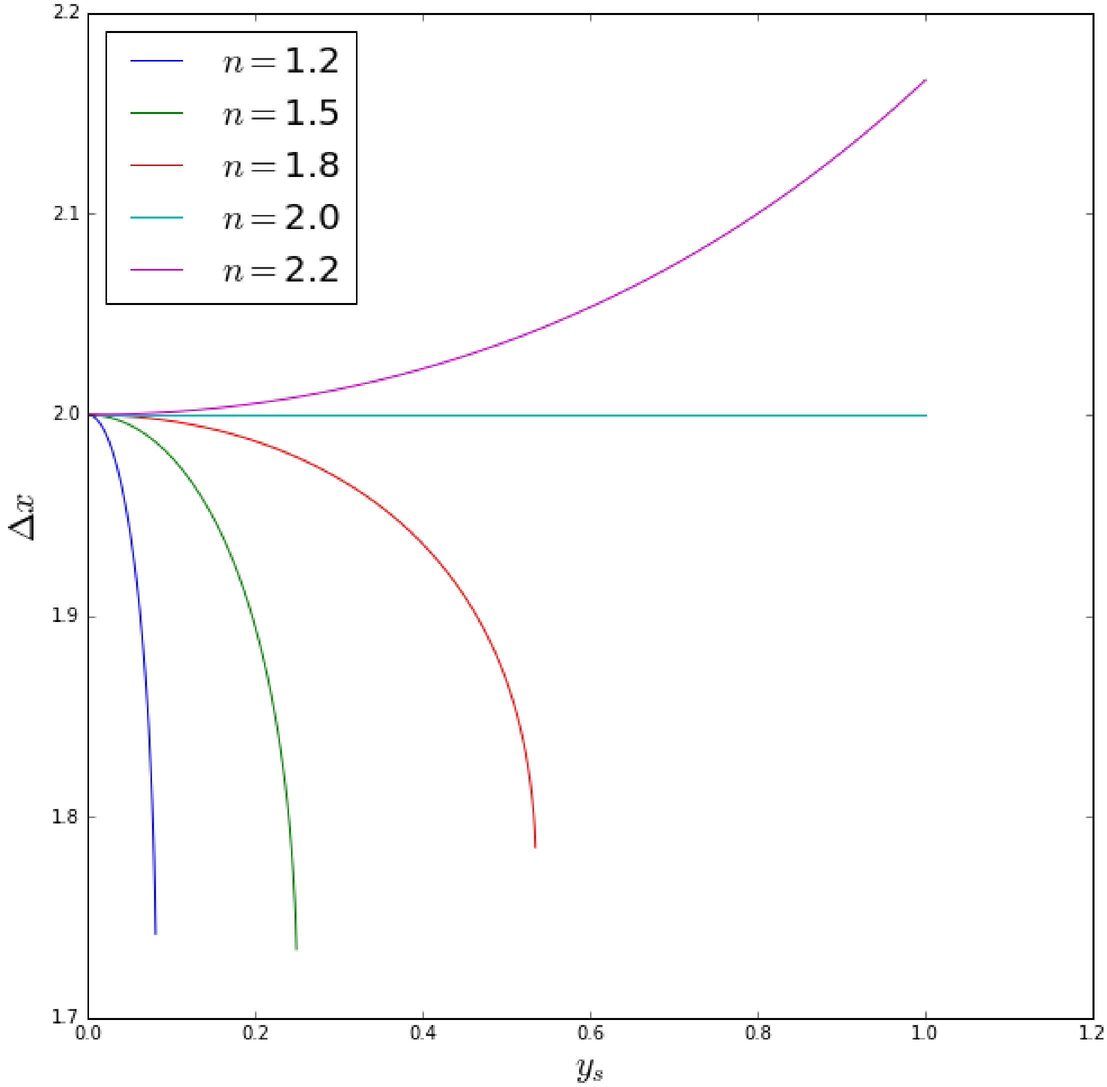
4 Image separation vs n

We explore now the sensitivity of image separation on the power-law index n.

```
In [33]: n=[1.2,1.5,1.8,2.0,2.2]

         x0=np.linspace(-3,-1.e-3,500)
         x1=np.linspace(1.e-3,3,500)

         ys=np.linspace(0,1,1000)

         fig,ax=plt.subplots(1,1,figsize=(10,10))

         for i in range(len(n)):
             a0=alpha(x0,n[i])
             a1=alpha(x1,n[i])
             s0=np.zeros(ys.size)
             s1=np.zeros(ys.size)
             s2=np.zeros(ys.size)
             dtheta=[]
             ytheta=[]
             te=[]
             for j in range(len(ys)):
                 def func(xx):
                     return f(xx,ys[j])-alpha(xx,n[i])
                 x0_ = fsolve(func,1.0)
                 x1_ = fsolve(func,-1.0)
                 x2_ = fsolve(func,0.0)
                 if (np.abs(func(x0_))<1e-8):
                     s0[j]=x0_[0]
                 if (np.abs(func(x1_))<1e-8 and x1_<0):
                     s1[j]=x1_[0]
                     dtheta.append(np.abs(s1[j]-s0[j]))
```

```
            ytheta.append(ys[j])
        if (np.abs(func(x2_))<1e-8):
            s2[j]=x2_[0]

    ax.set_xlim([0,1.2])
    ax.plot(ytheta,dtheta,'-',label='$n=$'+str(n[i]))
    ax.legend(loc='best',fontsize=20)
    ax.set_xlabel('$y_s$',fontsize=20)
    ax.set_ylabel('$\Delta x$',fontsize=20)
    ax.set_ylim([1.7,2.2])
```

```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:10: RuntimeWarning
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:10: RuntimeWarning
```



5 Time delays

The formula for the light travel time is

$$t(x) = \frac{(1+z_L)}{c}\frac{D_L D_S}{D_{LS}}\frac{\xi_0^2}{D_L^2}\left[\frac{1}{2}(x-y)^2 - \psi(x)\right] = \frac{(1+z_L)}{c}\frac{D_L D_S}{D_{LS}}\tau(x)$$

For a power-law lens, the lensing potential is

$$\psi(x) = \frac{1}{3-n}x^{3-n}$$

so that

$$\tau(x) = \frac{\xi_0^2}{D_L^2}\left[\frac{1}{2}(x-y)^2 - \frac{1}{3-n}x^{3-n}\right]$$

At the image positions

$$x - y = \alpha(x) = x^{2-n}$$

thus

$$\tau(x_i) = \frac{\xi_0^2}{D_L^2}\left[\frac{1}{2}x_i^{2(2-n)} - \frac{1}{3-n}x_i^{3-n}\right]$$

We can then compute the time delay between the images:

$$\Delta t_{ij} \propto \Delta\tau_{ij} = \frac{\xi_0^2}{D_L^2}\left[\frac{1}{2}\left(x_j^{2(2-n)} - x_i^{2(2-n)}\right) - \frac{1}{3-n}\left(x_j^{3-n} - x_i^{3-n}\right)\right]$$

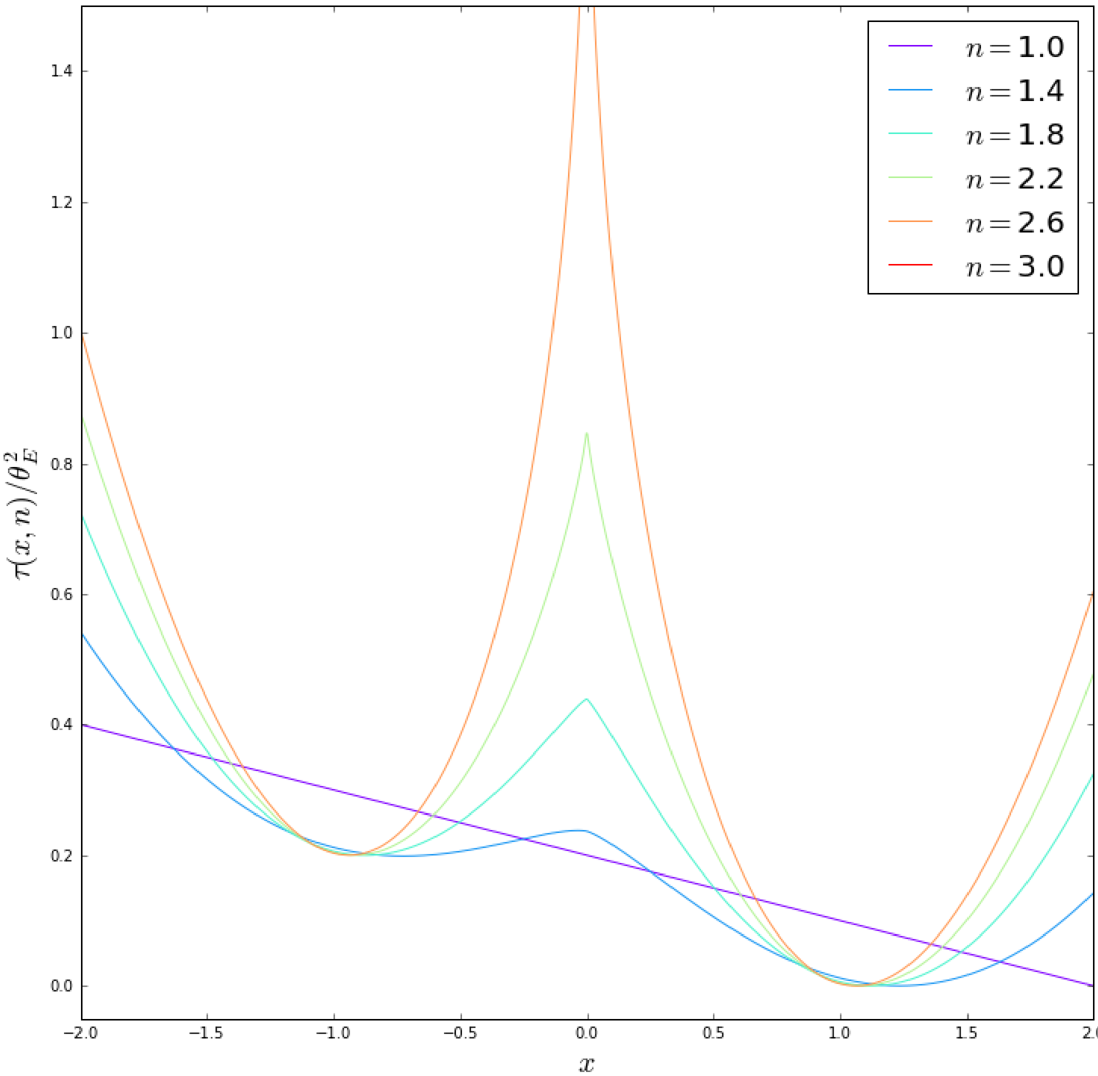
For $n = 2$, this formula gives:

$$\Delta\tau_{ij} = \frac{\xi_0^2}{D_L^2}(x_i - x_j) = \theta_E^2\left(\frac{\theta_i}{\theta_E} - \frac{\theta_j}{\theta_E}\right) = \frac{1}{2}\left(\theta_i^2 - \theta_j^2\right) = \Delta\tau_{SIS}$$

```
In [46]: def pot_power(x,n):
             return 1.0/(3.0-n)*np.abs(x)**(3.0-n)

         def travel_time_power(y,x,n):
             return (0.5*(x-y)**2-pot_power(x,n))

         x=np.linspace(-2.0,2.0)
         n=np.linspace(1.0,3.0,6)

         color=iter(cm.rainbow(np.linspace(0,1,n.size)))

         x=np.linspace(-2.0,2.0,1000)

         fig,ax=plt.subplots(1,1,figsize=(12,12))

         y=0.1
         for i in range(n.size):
             c=next(color)
             ax.plot(x,travel_time_power(y,x,n[i])-np.amin(travel_time_power(y,x,n[i])),color=c,label=r
             #ax.plot(x,pot_power(x,n[i]),color=c)
         ax.set_xlabel('$x$',fontsize=20)
         ax.set_ylabel(r'$\tau(x,n)/\theta_E^2$',fontsize=20)
         ax.legend(loc='best',fontsize=20)
         ax.set_ylim([-0.05,1.5])
```

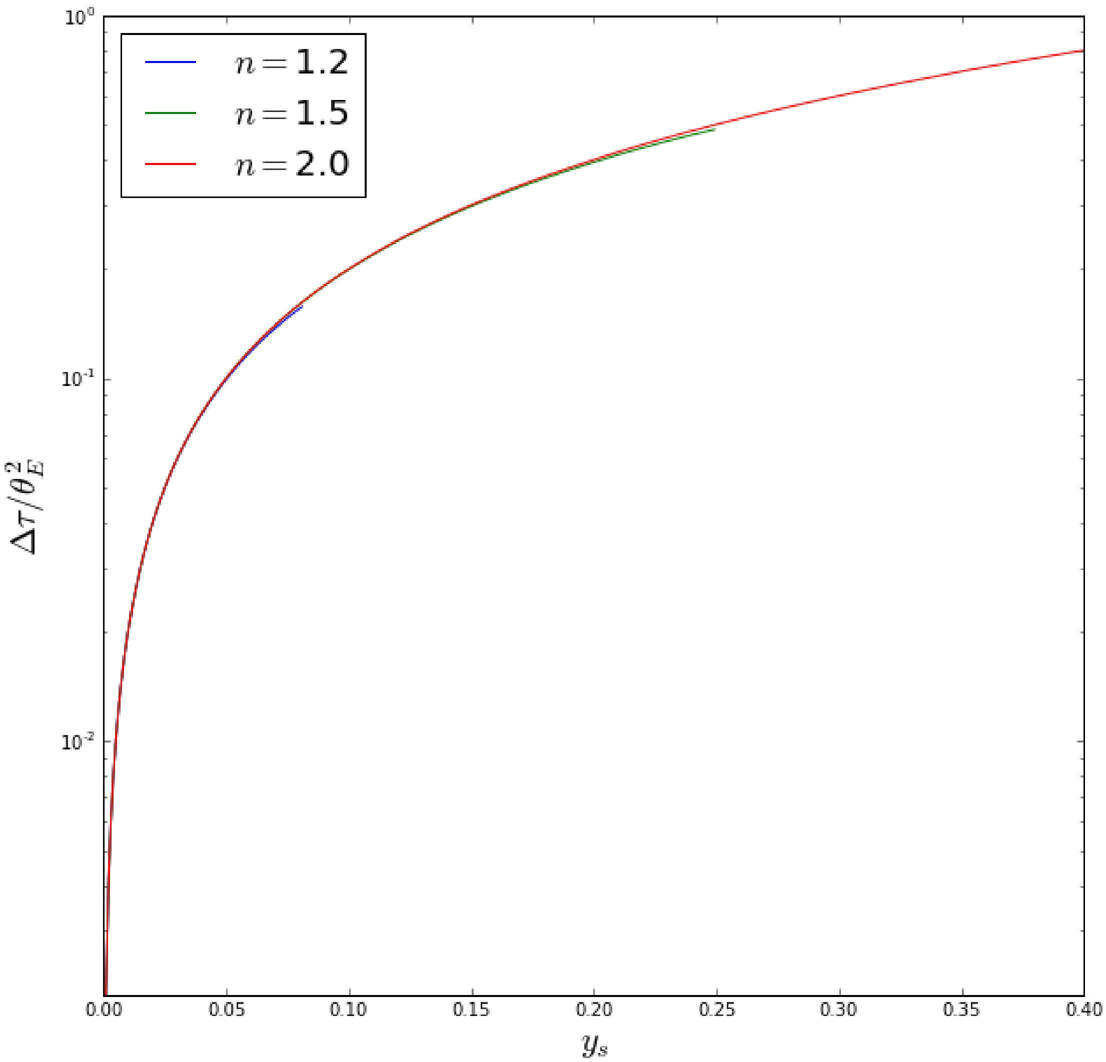
```
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:2: RuntimeWarning
  from ipykernel import kernelapp as app
/Users/massimo/anaconda/envs/python2/lib/python2.7/site-packages/ipykernel/__main__.py:19: RuntimeWarning
```

Out[46]: (-0.05, 1.5)



In [9]:
```
n=[1.2,1.5,2.0]

x0=np.linspace(-3,-1.e-3,500)
x1=np.linspace(1.e-3,3,500)

ys=np.linspace(0,1,1000)

fig,ax=plt.subplots(1,1,figsize=(10,10))

def time_delay(x1,x2,n):
    return(0.5*(x2**(2*(2-n))-x1**(2*(2-n)))-1.0/(3.0-n)*(x2**(3-n)-x1**(3-n)))
```

```
for i in range(len(n)):
    a0=alpha(x0,n[i])
    a1=alpha(x1,n[i])
    s0=np.zeros(ys.size)
    s1=np.zeros(ys.size)
    ytheta=[]
    td=[]
    dtheta=[]
    for j in range(len(ys)):
        def func(xx):
            return f(xx,ys[j])-alpha(xx,n[i])
        x0_ = fsolve(func,1.0)
        x1_ = fsolve(func,-1.0)
        if (np.abs(func(x0_))<1e-8):
            s0[j]=x0_[0]
        if (np.abs(func(x1_))<1e-8 and x1_<0):
            s1[j]=x1_[0]
            dtheta.append(s1[j]+s0[j])
            ytheta.append(ys[j])
            t=time_delay(s0[j],-s1[j],n[i])
            td.append(t)

    ax.set_xlim([0,0.4])
    ax.set_ylim([0,1.0])
    ax.set_yscale('log')
    ax.plot(ytheta,td,'-',label='$n=$'+str(n[i]))
    ax.legend(loc='best',fontsize=20)
    ax.set_xlabel('$y_s$',fontsize=20)
    ax.set_ylabel(r'$\Delta \tau/\theta_E^2$',fontsize=20)
```



In the case of small image separations with respect to the size of the Einstein radius (i.e. $y << 1$), Kochanek (2002) derived an approximated formula stating that

$$\Delta t(n) = (n-1)\Delta t_{SIS}\left[1 - \frac{(2-n)^2}{12}\left(\frac{\delta\theta}{\langle\theta\rangle}\right)^2 + ...\right]$$

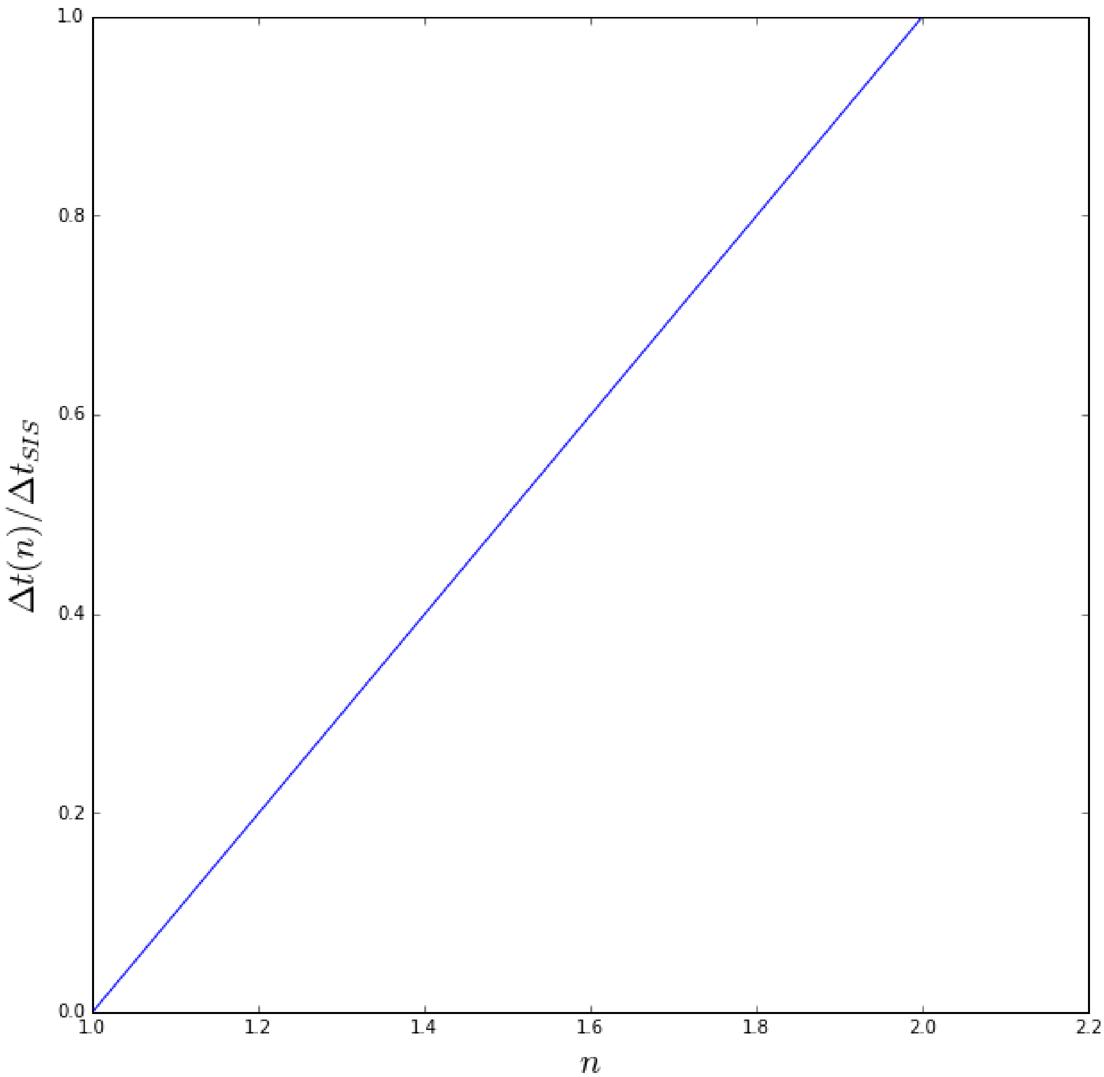
The formula however works also in the case of $\delta\theta/\langle\theta\rangle$ of order unity.

Let us assume to observe an image separation of $\delta\theta = 1$" and $\langle\theta\rangle = 5$". In this case $\Delta t(n)/\Delta t_{SIS}$ looks like this:

```
In [10]: def deltat(dtheta,thetam,n):
             return (n-1)*(1-(2-n)**2/12*dtheta**2/thetam**2)

         n=np.linspace(1.0,2.0,100)
         dt=deltat(1.0,5.0,n)

         fig,ax=plt.subplots(1,1,figsize=(10,10))
```

```
ax.plot(n,dt,'-')
ax.set_xlabel('$n$',fontsize=20)
ax.set_ylabel(r'$\Delta t(n)/\Delta t_{SIS}$',fontsize=20)
```

```
Out[10]: <matplotlib.text.Text at 0x11276f450>
```



This shows that more centrally concentrated lenses (larger n) produce larger time delays for a fixed image separation.